Part II Tripos Bookwork Solutions

FCM

1. Green's. $\oint_{\mathcal{C}} Ldx + Mdy = \iint_{\mathcal{D}} \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dxdy$. Stokes'. $\iint_{\mathcal{D}} \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial\mathcal{D}} \mathbf{F} \cdot d\mathbf{r}$.

2. $u(x,0) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{v(t,0)}{t-x}dt,\ v(x,0) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{u(t,0)}{t-x}dt.$

Define $\bar{f}(z) = \overline{f(\bar{z})}$, then $2u(x,y) = f(x+iy) + \bar{f}(x-iy)$.

Define $z = x+iy$, $w = x-iy$ then $2u(\frac{z+w}{2}, \frac{z-w}{2i}) = f(z) + \bar{f}(w) = f(z) + \overline{f(\bar{w})}$. Set $z_0 = \bar{w}$ const..

3. $\Gamma(z) = \lim\limits_{n\to\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)}$.

Define $\frac{1}{\Gamma_n(z)} = \frac{z(z+1)\cdots(z+n)}{n!n^z} = ze^{z(\sum\limits_{k=1}^{n}\frac{1}{k} - \log n)} \prod\limits_{s=1}^{n} e^{-\frac{z}{s}}(1+\frac{z}{s})$. Take limit.

4. $\Gamma(z) \coloneqq \int_0^\infty t^{z-1}e^{-t}dt,\ \mathrm{Re}z > 0.$

Identities:

$$\begin{aligned} \Gamma(z)\Gamma(1-z) &= \frac{\pi}{\sin \pi z} \text{ (Euler)} \\ \frac{2^{2z}\Gamma(z+\frac{1}{2})\Gamma(z)}{\Gamma(2z)} &= 2\sqrt{\pi} \text{ (Beta)} \\ \frac{1}{\Gamma(z)} &= \frac{1}{2\pi i}\int_{-\infty}^{0^+} t^{-z}e^t dt\ \forall z \in \mathbb{C}. \end{aligned}$$

$B(z_1, z_2) \coloneqq \int_0^1 x^{z_1-1}(1-x)^{z_2-1}dx,\ \mathrm{Re}z_1,\ \mathrm{Re}z_2 > 0.$

Identities: $B(z_1,z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}$.

$\zeta(s) \coloneqq \sum\limits_{n=1}^{\infty} n^{-s},\ \mathrm{Re}s > 1.$

Identities:

$$\begin{aligned}
\zeta(s) &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}}{e^t-1}dt,\ \mathrm{Re}s>1\ (\mathrm{Gamma}) \\
\zeta(s) &= \frac{\Gamma(1-s)}{2\pi i}\int_{-\infty}^{0^+} \frac{t^{s-1}}{e^{-t}-1}dt\ \forall s\in\mathbb{C}\backslash\{1\} \\
(s-1)\zeta(s) &= -\frac{\Gamma(2-s)}{2\pi i}\int_{-\infty}^{0^+} -t^{s-2}(1+\frac{t}{2}+\mathcal{O}(t^2))dt \\
\frac{2}{(2\pi)^s}\zeta(1-s) &= \zeta(s)\Gamma(s)\cos\frac{\pi s}{2}.
\end{aligned}$$

5. Elliptic functions are doubly periodic meromorphic functions.

Properties: 1) integral around a fundamental cell is zero; 2) no single simple poles; 3) if no poles then no poles in $\mathbb{C}$ and thus $\widetilde{\mathbb{C}}$; 4) $N-P=\oint_C \frac{f'(z)}{f(z)-c}dz=0$ by periodicity.

$\wp'(z)$ is odd and doubly periodic, so $\wp(z)-\mathrm{const.}$ is doubly periodic where $\mathrm{const.}=0$ by even-ness. Obtain

$$\wp'(z)^2-4\wp(z)^3+g_2\wp(z)+g_3=\mathcal{O}(z^2)$$

but the RHS is doubly periodic and analytic so by Liouville must be constant zero as $z\to 0$.

6. $z^4q(z),\ 2z-z^2p(z)$ bounded at infinity.

7. For hypergeometric equation $z(1-z)y''-[c-(a+b+1)]\,y'+aby=0$, the Papperitz symbol is

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{matrix}\right\}.$$

Also

$$\left(\frac{z-a}{z-b}\right)^k\left(\frac{z-c}{z-b}\right)^l P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\} = P\left\{\begin{matrix} T(a) & T(b) & T(c) & \\ \alpha+k & \beta-k-l & \gamma+l & T(z) \\ \alpha'+k & \beta'-k-l & \gamma'+l & \end{matrix}\right\}.$$

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{z^n}{n!}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(b)}\frac{\Gamma(c)}{\Gamma(c+n)}=\sum_{n=0}^{\infty}\frac{z^n}{n!}\frac{\Gamma(a+n)}{\Gamma(a)\Gamma(b)}\frac{\Gamma(c)}{\Gamma(c-b)}B(b+n,\ c-b).$$

Then use $(1-tz)^{-a}=\sum_{n=0}^{\infty}\frac{(zt)^n}{n!}\frac{\Gamma(a+n)}{\Gamma(a)}$, $\mathrm{Re}b,\ \mathrm{Re}(c-b)>0,\ |z|<1$.

CD

1. Given that the end points are fixed, the path taken by the system in configuration space is an extremum of the action.

2. Holonomic constraints are $f_\alpha(\mathbf{x},t)=0$ where $f_\alpha:\mathbb{R}\times\mathrm{C}\to\mathbb{R}$, $\alpha=1,2,\ldots,3N-n$. A one-parameter group of transformations $q_i(t)\mapsto Q_i(t,s)$ where $Q_i(t,0)=q_i(t)$ are a

continuous symmetry of the Lagrangian $\mathcal{L}$ if $\frac{\partial}{\partial s}\mathcal{L}(Q_i(t,s),\ \dot{Q}_i(t,s),t) = 0\,\forall t$. The associated conserved quantity is $\sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial Q_i}{\partial s}\Big|_{s=0}$.

3. $\mathbf{x} = \mathbf{x}_0$ is an equilibrium point for the equation of motion $\ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ if $\mathbf{f}(\mathbf{x}_0) = 0$. Perturbation $\mathbf{x} = \mathbf{x}_0 + \eta(t)$ satisfies $\ddot{\eta} = -F\eta$ where eigenvectors of F are normal modes.

4. $T = \frac{1}{2}\left\{\begin{array}{c}\sum_i m_i \\ \int d^3\mathbf{r}\rho(\mathbf{r})\end{array}\right\}\dot{\mathbf{r}}_i^2 = \frac{1}{2}\left\{\begin{array}{c}\sum_i m_i \\ \int d^3\mathbf{r}\rho(\mathbf{r})\end{array}\right\}\left(\omega_a\omega_a r_{(i)}^2 - r_{(i)_a}r_{(i)_b}\right) = \frac{1}{2}\omega_a I_{ab}\omega_b$, so

$$I_{ab} = \left\{\begin{array}{c}\sum_i m_i \\ \int d^3\mathbf{r}\rho(\mathbf{r})\end{array}\right\}\left(r^2\delta_{ab} - r_{(i)_a}r_{(i)_b}\right).$$

With $\mathbf{r} \mapsto \mathbf{r} - \mathbf{c}$, $I_{ab} \mapsto I_{ab} + M\left(c^2\delta_{ab} - c_a c_b\right)$.

5. $R_{ab} = \mathbf{e}_a \cdot \tilde{\mathbf{e}}_b$, $\mathbf{e}_a = R_{ab}\tilde{\mathbf{e}}_b$ where $\mathbf{r} = r_a\mathbf{e}_a(t) = \tilde{r}_a(t)\tilde{\mathbf{e}}_a$. $\frac{d\mathbf{e}_a}{dt} = \dot{R}_{ab}\left(R^{-1}\right)_{bc}\mathbf{e}_c$.
Define $\omega_{ab} = (\dot{R}R^{-1})_{ab}$. Have

$$\frac{d\mathbf{L}}{dt} = \dot{L}_c\mathbf{e}_c + L_a\omega_b\mathbf{e}_b \times \mathbf{e}_a = \dot{L}_c\mathbf{e}_c + \varepsilon_{cba}L_a\omega_b\mathbf{e}_c = 0$$

so $0 = \dot{L}_c + \varepsilon_{cba}L_a\omega_b$.

6. $\mathbf{e}_a = R_{ab}(\phi,\theta,\psi)\tilde{\mathbf{e}}_b$ where

$$\{\tilde{\mathbf{e}}_a\} \overset{R(\tilde{\mathbf{e}}_3,\phi)}{\longrightarrow} \{\mathbf{e}_a^{'}\} \overset{R(\mathbf{e}_1^{'},\theta)}{\longrightarrow} \{\mathbf{e}_a^{''}\} \overset{R(\mathbf{e}_3^{''},\psi)}{\longrightarrow} \{\mathbf{e}_a\}.$$

Have $\underline{\omega} = \phi\tilde{\mathbf{e}}_3 + \theta\mathbf{e}_1^{'} + \psi\mathbf{e}_3^{''}$ so

$$\underline{\omega} = \left(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right)\mathbf{e}_1 + \left(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\right)\mathbf{e}_2 + \left(\dot{\psi} + \dot{\phi}\cos\theta\right)\mathbf{e}_3.$$

You should be able to draw the diagram.

7. $K\mathcal{J}K^T = \mathcal{J}$ where $\mathcal{J} = \begin{pmatrix}0 & I \\ -I & 0\end{pmatrix}$ and K is the Jacobian.

8. The action-angle variables are a canonical transformation $(q,p) \mapsto (\theta, I)$ s.t. $H(q,p) = \tilde{H}(I)$, in which case by Hamilton's equations the system can be integrated up.

$$I = \frac{1}{2\pi}\oint_\Gamma p\,dq.$$

9. A function $I(p,q,\lambda)$ is an adiabatic invariant if $|I(t) - I(0)| = \mathcal{O}(\epsilon)\,\forall t : 0 < t < \frac{T}{\epsilon}$ where T is the period of the parameter λ. The action variable is an example.

C

1. Homogeneity and isotropy. $\mathbf{v}_B^A = \mathbf{v}_B^O - \mathbf{v}_A^O$ but velocities must be a function of relative positions so $\mathbf{v}(\mathbf{r}_B - \mathbf{r}_A) = \mathbf{v}(\mathbf{r}_B) - \mathbf{v}(\mathbf{r}_A)$, i.e. velocity is linearly related to relative position $\mathbf{v} = H\mathbf{r}$.

2. $\gtrsim 10^{10}\,\mathrm{K}$, $\{\mathrm{e}^{\pm}, \gamma, \nu \text{ and } \bar{\nu}\}$; $5 \times 10^9 \sim 10^{10}\,\mathrm{K}$, $\{\mathrm{e}^{\pm}, \gamma\}$; $\lesssim 5 \times 10^9\,\mathrm{K}$, $\{\gamma\}$.

3. Matter-dominated. $z \ll z_{\mathrm{eq}}$, $a \propto t^{\frac{2}{3}}$, $\rho = \frac{1}{6\pi G t^2}$, $T \propto t^{-\frac{2}{3}}$.

Radiation-dominated. $z \gg z_{\mathrm{eq}}$, $a^{\frac{1}{2}}$, $\rho = \frac{3}{32\pi G t^2}$, $T \sim \frac{10^{10}\,\mathrm{K}}{t^{\frac{1}{2}}}$.

Key. $a \propto t^{\frac{2}{3\gamma}}$, $\rho = \frac{1}{6\pi G \gamma^2 t^2}$.

4. $^4\mathrm{He}$ does not depend sensitively on ρ_{baryon} but D and $^3\mathrm{H}$ do.

5.

$$\dot{n} + \underbrace{3\frac{\dot{a}}{a}n}_{\text{depletion due to expansion}} = \underbrace{-g_i \langle \sigma v \rangle n\bar{n}}_{\text{annihilation}} + \underbrace{\Pi(t)}_{\text{production}} .$$

At equilibrium, relax to $H = 0$, $n = \bar{n}$, $\dot{n} = 0$ so

$$\Pi(t) = g_i \langle \sigma v \rangle n_{\mathrm{eq}}^2$$

and

$$\dot{n} + 3Hn = g_i \langle \sigma v \rangle \left(n_{\mathrm{eq}}^2 - n^2 \right) .$$

Change of variable gives

$$\frac{dY}{dx} = -\frac{\lambda}{x^2} \left(Y^2 - Y_{\mathrm{eq}}^2 \right)$$

where λ is a const.. Now $x \gg x_\star$, $Y \gg Y_{\mathrm{eq}}$, $Y_\star \gg Y_\infty$.

6. Flatness. $k = 0$ solution is unstable, but the observed universe has very low curvature (or low energy density associated with curvature, or close to critical density so 'fine-tuning').

Horizon. Remarkable homogeneity and isotropy in regions with no possible causal contact due to finite light speed.

Inflation. A finite period of accelerated expansion.

Conditions: $\ddot{a} > 0$ requires $3\gamma < 2 \Leftrightarrow \rho + 3P < 0$ so density falls slower than curvature term. Accelerated expansion means regions far apart could have been in causal contact.

7. $\ddot{\phi} \ll \left| 3H\dot{\phi} \right|$, $V'(\phi)$ and $\frac{1}{2}\dot{\phi}^2 \ll V$. e-fold number

$$N = \ln \frac{a_{\mathrm{e}}}{a_{\mathrm{i}}} = \int_{a_{\mathrm{i}}}^{a_{\mathrm{e}}} d\ln a = \int_{t_{\mathrm{i}}}^{t_{\mathrm{e}}} H dt.$$

8. Linearised $\frac{\partial \rho}{\partial t} = -\rho_0 \nabla \cdot \mathbf{v},\ \frac{\partial \mathbf{v}}{\partial t} + \frac{c_s^2}{\rho_0}\nabla\rho_1 + \nabla\Phi_1 = 0,\ \nabla^2\Phi_1 = 4\pi G\rho_0.$

AM

1. Suppose $g(t) \sim t^\alpha \sum_{j=0}^{\infty} a_j t^{rj}$, $|g(t)| \leq Ke^{bt}$ some K, $b > 0$, then for $\alpha > -1$,

$$\int_0^\infty e^{-xt} g(t)dt \quad \sim \quad \sum_{j=0}^{\infty} a_j \frac{\Gamma(\alpha + rj + 1)}{x^{\alpha+rj+1}}$$

as $x \to \infty$.

2. $f \sim g$ as $x \to x_0$ if $f(x) = o(g(x))$.

$\{\phi_n(x)\}_{n=0}^{\infty}$ is an asymptotic sequence if as $x \to x_0$, $\phi_{n+1}(x) = o\left(\phi_n(x)\right)\ \forall n$.

$f \sim \sum_{n=0}^{\infty} a_n\phi_n$ if $\{\phi_n\}$ is an asymptotic sequence as $x \to x_0$ and $f - \sum_{n=0}^{N} a_n\phi_n = o(\phi_N)\ \forall N$.

Uniqueness.

$$\begin{aligned} a_0 &= \lim_{x \to x_0} \frac{f(x)}{\phi_0(x)} \\ a_n &= \lim_{x \to x_0} \frac{f(x) - \sum_{i=1}^{n-1} \phi_i(x)}{\phi_n(x)}. \end{aligned}$$

Optimal truncation. Truncate the series at $n = N_x$ s.t. the first excluded term has the least magnitude.

3. Recall $\epsilon^2 \frac{d^2 y}{dx^2} = Q(x)y$ has Liouville-Green approximations

$$\begin{array}{c} y \sim A_\pm Q(x)^{-\frac{1}{4}} e^{\pm\frac{1}{\epsilon}\int^x \sqrt{Q(x')}dx'},\ Q > 0 \\ y \sim A_\pm \left|Q(x)\right|^{-\frac{1}{4}} e^{\pm\frac{i}{\epsilon}\int^x \sqrt{|Q(x')|}dx'},\ Q < 0 \end{array}.$$

So for the Schrödinger equation, set $Q(x) = V(x) - E$, then

$$\begin{aligned} x < x_1,\quad y \sim &\quad A\left(V(x) - E\right)^{-1/4} \exp\left(-\frac{1}{\epsilon}\int_x^{x_1} \sqrt{V(x') - E}dx'\right) \\ x_1 < x < x_2,\quad y \sim &\quad 2A\left|V(x) - E\right|^{-1/4} \sin\left(\frac{1}{\epsilon}\int_{x_1}^{x} \sqrt{|V(x') - E|}dx' + \frac{\pi}{4}\right) \\ x > x_2,\quad y \sim &\quad 2B\left(V(x) - E\right)^{-1/4} \exp\left(-\frac{1}{\epsilon}\int_{x_2}^{x} \sqrt{V(x') - E}dx'\right) \\ x_1 < x < x_2,\quad y \sim &\quad B\left|V(x) - E\right|^{-1/4} \sin\left(\frac{1}{\epsilon}\int_{x}^{x_2} \sqrt{|V(x') - E|}dx' + \frac{\pi}{4}\right) \end{aligned}$$

but matching phase means

$$\sin\left(\frac{1}{\epsilon}\int_x^{x_2}\sqrt{|V(x')-E|}dx'+\frac{\pi}{4}\right) = \sin\left(-\frac{1}{\epsilon}\int_{x_1}^{x}\sqrt{|V(x')-E|}dx'-\frac{\pi}{4}+\triangle+\frac{\pi}{2}\right)$$

so

$$\triangle \equiv \frac{1}{\epsilon}\int_{x_1}^{x_2}\sqrt{|V(x)-E|}dx = (n+\frac{1}{2})\pi.$$

For $y'' = -EQ(x)y$ on interval $[0, L]$ where E, $Q > 0$ and boundary conditions are set $y(0) = y(L) = 0$, have

$$y \sim E^{-1/4}Q(x)^{-1/4}\left[a\sin\left(\int_0^x\sqrt{EQ(x')}dx'\right)+b\cos\left(\int_0^x\sqrt{EQ(x')}dx'\right)\right]$$

and by b.c. have $b = 0$ and

$$E^{\mathrm{LG}} = \left(\frac{n\pi}{\int_0^L\sqrt{Q(x)}dx}\right)^2.$$

IS

1. A flow map $\mathbf{x} = g^\epsilon\mathbf{x}_0$ is generated by a vector field $\mathbf{V}$ if $\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0$. It has properties: $g^0 = \iota$, $g^t g^s = g^{t+s}$ and $g^{-t} = (g^t)^{-1}$.

$$[\mathbf{V}_1,\, \mathbf{V}_2] = (\mathbf{V}_1\cdot\partial_{\mathbf{x}})\,\mathbf{V}_2 - (\mathbf{V}_1\cdot\partial_{\mathbf{x}})\,\mathbf{V}_2.$$

$$\{f,\, g\} = \frac{\partial f}{\partial\mathbf{x}}\cdot J\frac{\partial g}{\partial\mathbf{x}} = \mathbf{v}_g\cdot\partial_{\mathbf{x}}f.$$

A Hamiltonian system $\dot{\mathbf{x}} = J\frac{\partial H}{\partial\mathbf{x}} = \mathbf{v}_H(\mathbf{x})$. Now $[\mathbf{v}_f, \mathbf{v}_g] = -\mathbf{v}_{\{f,g\}}$ shown by acting $.\partial_{\mathbf{x}}h$ and using the Jacobi identity.

2. Scattering data $S = \bar{S}\cup\{T(k)\}$ where $\bar{S} = \big\{\{\chi_n, c_n(t)\}_{n=1}^N, R(k)\big\}$.

Define

$$F(x;t) = \sum_{n=1}^N c_n^2(t)e^{-\chi_n x} + \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}R(k)dk$$

and $K(x, y)$ to be the unique solution to

$$F(x+y) + K(x,y) + \int_x^\infty K(x,z)F(y+z)dz = 0.$$

Then $u(x) = -2\frac{d}{dx}K(x, x)$.

Reflectionless. $F(x;t) = \sum_{n=1}^{N} c_n^2(t)e^{-\chi_n x}$ and must have the form

$$K(x,y) = \sum_{n=1}^{N} K_n(x)e^{-\chi_n y}.$$

Substitution yields

$$A\mathbf{K} = -\mathbf{c}$$

where

$$\begin{aligned} \mathbf{K} &= (K_1(x), \ldots, K_N(x))^T \\ \mathbf{c} &= \left(c_1^2 e^{-\chi_1 x}, \ldots, c_N^2 e^{-\chi_N x}\right) \\ A_{nm} &= \delta_{nm} + c_n^2 \frac{e^{-(\chi_n+\chi_m)x}}{\chi_n + \chi_m} \\ A'_{nm} &= -c_n^2 e^{-(\chi_n+\chi_m)x}. \end{aligned}$$

Hence

$$K(x,\, x) \;=\; \sum_{m=1}^{N}\sum_{n=1}^{N} (A^{-1})_{mn} A'_{nm} = \operatorname{tr}\left(A^{-1}A\right) = (\log\det A)'$$

and

$$u(x,\, t) = -2\partial_x^2 \log\det A.$$

3. $\frac{dL}{dt} = [L, A]$ so $\lambda_t\psi = L_t\psi + L\psi_t - \lambda\psi_t = (L-\lambda)(\psi_t + A\psi)$. Now $0 \le \lambda_t \left\|\psi\right\|^2 = \langle (L-\lambda)\psi, \psi_t + A\psi\rangle = 0$ so $\lambda_t = 0$ and $L\psi' = \lambda\psi'$.

4. An evolution equation is Hamiltonian form if written $u_t = \mathcal{J}\delta H$ for some functional $H \equiv H[u]$ and antisymmetric linear operator $\mathcal{J}$ s.t. $\{F, G\} = \langle \delta F, \mathcal{J}\delta G\rangle$ defines a Poisson bracket.

$F[u + \epsilon\eta] = F[u] + \epsilon \langle \delta F, \eta\rangle + o(\epsilon)$ where

$$F[u] = \int f(x, u, u_x, \ldots, u^{(n)})dx$$

where $\eta^{(k)} \to 0$ as $|x| \to \infty$, so

$$\delta F = \frac{\partial f}{\partial u} - D_x \frac{\partial f}{\partial u_x} + D_x^2 \frac{\partial f}{\partial u_{xx}} - \cdots .$$

5. $I[u] = \int \iota(x, u, u_x, \cdots)dx$ gives

$$\begin{aligned} \dot{I}[u] &= \int \left(u_t \frac{\partial \iota}{\partial u} + u_{xt} \frac{\partial \iota}{\partial u_x} + \cdots \right) dx \\ &= \int u_t \left(\frac{\partial \iota}{\partial u} - D_x \frac{\partial \iota}{\partial u_x} + \cdots \right) dx \\ &= \langle \mathcal{J} \delta H, \delta I \rangle \\ &= \{I, H\}. \end{aligned}$$

6. Given $\triangle[x, u, u_x, \ldots, u^{(n)}] = 0$, $g^\epsilon : (x, u) \mapsto (\tilde{x}, \tilde{u})$ is a Lie point symmetry if

$$\triangle \mathrm{pr}^{(n)} g^\epsilon [x, u, u_x, \ldots, u^{(n)}] = 0$$

where $\mathrm{pr}^{(n)} g^\epsilon : [x, u, u_x, \ldots, u^{(n)}] \mapsto [\tilde{x}, \tilde{u}, \tilde{u}_{\tilde{x}}, \ldots, \tilde{u}^{(n)}]$ is the n-th prolongation. Equivalently, $\mathrm{pr}^{(n)} V \triangle = 0$.

$\mathbf{V} = \frac{d}{d\epsilon}|_{\epsilon=0} g^\epsilon \mathbf{x}$. Common transformations are $V = \alpha \partial_{x_i} \Rightarrow \tilde{x}_i = x_i + \alpha\epsilon$, $V = \beta x_i \partial_{x_i} \Rightarrow \tilde{x}_i = e^{\beta\epsilon} x_i$.

For

$$\begin{aligned} \tilde{x} &= x + \epsilon \xi(x, u) + o(\epsilon) \\ \tilde{u} &= u + \epsilon \eta(x, u) + o(\epsilon) \\ \tilde{u}^{(k)} &= u^{(k)} + \epsilon \eta_k + o(\epsilon), \\ V &= \xi(x, u) \frac{\partial}{\partial x} + \eta(x, u) \frac{\partial}{\partial u} \\ \mathrm{pr}^{(n)} V &= V + \sum_{k=1}^{n} \eta_k \partial_{u^{(k)}} \end{aligned}$$

have prolongation formula

$$\eta_{k+1} = D_x \eta_k - u^{(k+1)} D_x \xi.$$

To see this, use the contact condition $d\tilde{u}^k = \frac{d\tilde{u}^k}{dx} dx$, so

$$\begin{aligned} \mathrm{LHS} &= \left(u^{(k+1)} + \epsilon D_x \eta_k\right) dx + o(\epsilon) \\ \mathrm{RHS} &= \tilde{u}^{(k+1)} \left(1 + \epsilon D_x \xi\right) dx + o(\epsilon) \end{aligned}$$

and $\tilde{u}^{(k+1)} = \left(u^{(k+1)} + \epsilon D_x \eta_k\right)\left(1 - \epsilon D_x \xi\right) + o(\epsilon) = u^{(k+1)} + \epsilon\left(D_x \eta_k - u^{(k+1)} D_x \xi\right) + o(\epsilon)$.

Finally:

$\Rightarrow$: $0 = \frac{d}{d\epsilon}\Big|_{\epsilon=0} g^\epsilon \mathbf{x} \cdot \frac{\partial F}{\partial \mathbf{x}}(g^\epsilon \mathbf{x})\Big|_{\epsilon=0} = VF$.

$\Leftarrow$: Let $S = \{\mathbf{x} : F(\mathbf{x}) = 0\}$ then $VF = 0 \Rightarrow \mathbf{V}$ is tangent to surface S. Hence g^ϵ maps S to itself, i.e. $F(\mathbf{x}) = 0$ implies $F(g^\epsilon \mathbf{x}) = 0$.

PQM

1. $\langle x|p\rangle = \frac{e^{ixp/\hbar}}{\sqrt{2\pi\hbar}}$, $\langle p|x\rangle = \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}}$, $\langle x|x'\rangle = \delta(x-x')$, $\langle p|p'\rangle = \delta(p-p')$.

Completeness. $|\psi\rangle = \sum_n |\chi_n\rangle\langle\chi_n|\psi\rangle$ so $\sum_n |\chi_n\rangle\langle\chi_n| = \mathbb{I}$ etc. Hence $|\psi\rangle = \int dx\psi(x)|x\rangle = \int dp\tilde{\psi}(p)|p\rangle$.

Further, $\langle x|\hat{p}|x'\rangle = -i\hbar\partial_x\delta(x-x')$, $\langle p|\hat{x}|p'\rangle = i\hbar\partial_p\delta(p-p')$, and

$$\begin{aligned}\langle p|V(\hat{x})|\psi\rangle &= \int dx \int dp' \frac{1}{2\pi\hbar} e^{-i(p-p')x/\hbar} V(x)\tilde{\psi}(p') \\ &= \int dp' \frac{1}{\sqrt{2\pi\hbar}} \tilde{V}(p-p')\tilde{\psi}(p')\end{aligned}$$

so the Schrödinger equation becomes in momentum space

$$\frac{p^2}{2m}\tilde{\psi}(p) + \frac{1}{\sqrt{2\pi\hbar}}\int dp'\tilde{V}(p-p')\tilde{\psi}(p') = E\tilde{\psi}(p).$$

2. $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, $a|n\rangle = \sqrt{n}|n-1\rangle$, $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$.

3. Notation. $H = H_0 + \epsilon V$.

Non-degenerate case.

$$\begin{aligned}E &= E_n^{(0)} + \epsilon\langle n|V|n\rangle + \epsilon^2\sum_{j\neq n}\frac{|\langle j|V|n\rangle|^2}{E_n^{(0)} - E_j^{(0)}} + \mathcal{O}(\epsilon^3) \\ |\psi\rangle &= N\left(|n\rangle + \epsilon\sum_{j\neq n}\frac{\langle j|V|n\rangle}{E_n^{(0)} - E_j^{(0)}}|j\rangle + \mathcal{O}(\epsilon^2)\right)\end{aligned}$$

where $N = 1 + \mathcal{O}(\epsilon)$ is an overall normalisation constant, and superscript $^{(0)}$ denotes the unperturbed eigen-energy for $|n\rangle$ and $|j\rangle_{j\neq n}$.

Degenerate case. $E^{(1)}$ is the eigenvalue of the matrix $V_{sr} = \langle s|V|r\rangle$ where $|r\rangle$, $|s\rangle$ are eigenvectors in the degenerate eigenspace V_λ. There are typically $\dim V_\lambda$ solutions.

4. $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Have $\mathbf{S} = \frac{\hbar}{2}\sigma$.

Properties: 1) $\sigma_i\sigma_j = \delta_{ij}\mathbb{I} + i\hbar\varepsilon_{ijk}\sigma_k$; 2) $\sigma_i\sigma_i = \mathbb{I}$ (no s.c.); 3) $(\mathbf{a}\cdot\sigma)(\mathbf{b}\cdot\sigma) = (\mathbf{a}\cdot\mathbf{b})\mathbb{I} + i(\mathbf{a}\times\mathbf{b})\cdot\sigma$.

5. $-j \leq m \leq j$ and $j \pm m$ are integers, so $j \in \mathbb{Z}$ (orbital) or $\mathbb{Z} + \frac{1}{2}$ (spin). Note there are $2j+1$ steps in between. $J_\pm J_\mp = \mathbf{J}^2 - J_3^2 \pm \hbar J_3$.

6. Note there are $\sum_{j=|j_1-j_2|}^{j_1+j_2}(2j+1) = (2j_1+1)(2j_2+1)$ states.

7. A unitary transformation U is a symmetry if $[U,\ H] = 0 \Leftrightarrow U^\dagger H U = H$.

8. The time evolution operator is $U(t) = e^{-iH_0 t/\hbar}$.

Interaction picture. $|\overline{\psi(t)}\rangle = e^{iH_0 t/\hbar}|\psi(t)\rangle$, $\overline{V(t)} = e^{iH_0 t/\hbar}V(t)e^{-iH_0 t/\hbar}$, $H = H_0 + \epsilon V(t)$

Equation of motion. $i\hbar\partial_t|\overline{\psi(t)}\rangle = \epsilon\overline{V(t)}|\overline{\psi(t)}\rangle$.

Integral equation. $|\overline{\psi(t)}\rangle = |\psi(0)\rangle - \frac{i\epsilon}{\hbar}\int_0^t dt'\overline{V(t')}|\psi(0)\rangle + \mathcal{O}(\epsilon^2)$.

Transition rate. Assuming $\langle f|i\rangle = 0$,

$$P(t) = \frac{\epsilon^2}{\hbar^2}\left|\int_0^t dt' e^{it'\frac{E_f - E_i}{\hbar}}\langle f|V|i\rangle\right|^2 + \mathcal{O}(\epsilon^3).$$

9. For joint space $V = U \otimes W$, $|\Psi\rangle = |\psi\rangle \otimes |\phi\rangle$, the (reduced) density operator is $\rho = |\Psi\rangle\langle\Psi|$ ($\bar{\rho} = |\psi\rangle\langle\psi|$). Have for $Q = Q \otimes \mathbb{I}$,

$$\begin{aligned} \langle Q\rangle_\Psi &= \mathrm{tr}(Q\rho) \\ \langle Q\rangle &= \mathrm{tr}(Q\bar{\rho}) \end{aligned}$$

where $\rho = \mathrm{diag}(p_i)$ when diagonalised.

FD II

1. On physical grounds, stress is symmetric and includes an isotropic pressure term and a deviatoric term that is instantaneous and linear in $\nabla\mathbf{u}$: $\sigma_{ij} = -p\delta_{ij} + \sigma_{ij}^{\mathrm{dev}}(\nabla\mathbf{u})$.

Suppose $\sigma_{ij}^{\mathrm{dev}} = A_{ijkl}\frac{\partial u_k}{\partial x_l}$ where rank four tensor A must be isotropic:

$$A_{ijkl} = \mu_1\delta_{ij}\delta_{kl} + \mu_2\delta_{ik}\delta_{jl} + \mu_3\delta_{il}\delta_{jk}$$

then $\sigma_{ij}^{\mathrm{dev}} = 2\mu e_{ij}$ by incompressibility and symmetry.

2.

$$\underbrace{\frac{d}{dt}\int_V \frac{1}{2}\rho\mathbf{u}^2 dV}_{\text{rate of change of kinetic energy}} + \underbrace{\int_{\partial V}\frac{1}{2}\rho\mathbf{u}^2\mathbf{u}\cdot\mathbf{n}dS}_{\text{kinetic energy flux over boundary}} =$$

$$\underbrace{\int \rho\mathbf{u}\cdot\mathbf{F}dV}_{\text{power against body forces}} + \underbrace{\int_{\partial V}\mathbf{u}\cdot\sigma\cdot\mathbf{n}dS}_{\text{power against surface tractions}} - \underbrace{\int_V(\sigma\cdot\nabla)\cdot\mathbf{u}dV}_{\text{dissipation}}$$

where $\sigma_{ij}\partial_j u_i = -p\nabla\cdot\mathbf{u} + 2\mu e_{ij}(e_{ij} + \Omega_{ij}) = 2\mu e_{ij}e_{ij} =: \Phi$ by incompressibility and anti-symmetry.

3. Instantaneity: instantaneous response to forces and boundary conditions. Linearity(:) of the velocity and pressure fields to forces and boundary conditions. Spatial and temporal reversibility.

4. $\int_V 2\mu e^s_{ij} e_{ij} dV = \int_V (\sigma^s_{ij} + p^s\delta_{ij})(\frac{\partial u_i}{\partial x_j} - \Omega_{ij})dV = \int_V \sigma^s_{ij}\partial_j u_i dV = \int_S \sigma^s_{ij} n_j u_i dS$ by incompressibility, antisymmetry and Stokes' equation.

Minimality.$\int_V 2\mu e^s_{ij} e^s_{ij} dV \le \int_V 2\mu e_{ij} e_{ij} dV$ for a Stokes flow $\mathbf{u}^s$ and an admissable flow $\mathbf{u}$ satisfying the same boundary condition.

$\int_V 2\mu e_{ij} e_{ij} dV = \int_V 2\mu e^s_{ij} e^s_{ij} dV + \underbrace{\int_V 2\mu \left(e_{ij} - e^s_{ij}\right)\left(e_{ij} - e^s_{ij}\right) dV}_{\ge 0} + \underbrace{\int_V 4\mu e^s_{ij}\left(e_{ij} - e^s_{ij}\right)}_{2\int_S \sigma^s_{ij} n_j (u_i - u^s_i) dS = 0}.$

Reciprocity. Set $\mathbf{u}^s$ and $\mathbf{u}$ to be two Stokes flows $\mathbf{u}^{(1),(2)}$, then $\int_S \sigma^{(1)}_{ij} n_j u^{(2)}_i dS = \int_S \sigma^{(2)}_{ij} n_j u^{(1)}_i dS$. Consider rigid body motion $\mathbf{u}^{(1),(2)}\big|_S = \mathbf{U}^{(1),(2)} + \mathbf{\Omega}^{(1),(2)} \times \mathbf{x}$, then

$$-\mathbf{U}^{(1)} \cdot \mathbf{F}^{(2)} - \mathbf{\Omega}^{(1)} \cdot \mathbf{G}^{(2)} = -\mathbf{U}^{(1)} \cdot \mathbf{F}^{(2)} - \mathbf{\Omega}^{(1)} \cdot \mathbf{G}^{(2)}$$

since $\int_S \sigma^{(1)}_{ij} n_j \left(\mathbf{\Omega}^{(2)} \times \mathbf{x}\right)_i dS = \mathbf{\Omega}^{(2)} \cdot \int_S \mathbf{x} \times \sigma^{(1)} \cdot \mathbf{n} dS$.

5. Assumptions. $\mathbf{u} = [u(y,\ z,\ t), 0, 0]$ by incompressibility, $\mathbf{F} = \mathbf{0}$, $p = p(x,\ t)$.

Equation. $\rho\frac{\partial u}{\partial t} = -\frac{dp}{dx} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$ where by variable dependence $p_x = \text{const.}$.

6. Assumptions. $h \ll L$ and rRe $\ll 1$. Pressure gradient balances viscosity.

Simplification.

$$\underbrace{\frac{\partial u}{\partial x}}_{u/L} \underset{\sim}{+} \underbrace{\frac{\partial v}{\partial y}}_{v/h} = 0 \Rightarrow v = \frac{h}{L}u \ll u$$

$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + \underbrace{(\mathbf{u}\cdot\nabla)\,u}_{u^2/L \sim uv/h}\right) = -\underbrace{p_x}_{p/L} + \mu\left(\underbrace{\cancel{\frac{\partial^2}{\partial x^2}}}_{1/L^2} \underset{\ll}{+} \underbrace{\frac{\partial^2}{\partial y^2}}_{1/h^2}\right) u$$

so $p \sim \frac{\mu u L}{h^2}$ and $\frac{\text{LHS}}{\text{RHS}} = \underbrace{\frac{\rho u h}{\mu}}_{\text{Re}}\frac{h}{L}$.

$$\underbrace{\rho\left(\mathbf{u}\cdot\nabla\right)v}_{\rho u v/L \sim \rho v^2/h} \underset{\ll}{=} -\underbrace{p_y}_{\mu u L/h^3} \underset{\gg}{+} \mu \underbrace{\frac{\partial^2 v}{\partial y^2}}_{\mu v/h^2}$$

so $p = p(x)$.

Equations. $u_x + v_y = 0, 0 = -\frac{dp}{dx} + \mu\frac{\partial^2 u}{\partial y^2}$.

7. Assumptions. Adopt Euler's limit outside the boundary layer. Pressure field set by external flow to the boundary layer. Inside the layer viscosity is balanced by inertia.

Simplification.

$$\underbrace{\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)}_{\rho u^2/L}\underset{\sim}{=}-p_x+\mu\left(\underbrace{\frac{\partial^2}{\partial x^2}}_{1/L^2}\ll+\underbrace{\frac{\partial^2}{\partial y^2}}_{1/\delta^2}\right)u$$

so $\delta=\sqrt{\frac{\nu L}{U}}$, $\Delta p_x\sim\rho U^2$.

$$\underbrace{\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)}_{\rho uv/L\sim\rho v^2/\delta}\underset{\sim}{=}-\underbrace{p_y}_{\Delta p_y/\delta}+\underbrace{\mu\frac{\partial^2 v}{\partial y^2}}_{\mu v/\delta^2}$$

so $\Delta p_y\sim\rho U^2\left(\frac{\delta}{L}\right)^2\ll\Delta p_x$.

Equations. $u_x+v_y=0$ and

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\rho\left(\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}\right)+\mu\frac{\partial^2 u}{\partial y^2}$$

subject to $u\to U$ as $y/\delta\to\infty$.

8. $\nabla\times$ N.S. where written $(\mathbf{u}\cdot\nabla)\,\mathbf{u}=\nabla\frac{1}{2}\mathbf{u}^2-\mathbf{u}\times(\nabla\times\mathbf{u})$,

$$\rho\left[\frac{\partial\underline{\omega}}{\partial t}-\nabla\times(\mathbf{u}\times\underline{\omega})\right]=\mu\nabla^2\underline{\omega}$$

but then

$$\frac{\partial\underline{\omega}}{\partial t}+\underbrace{(\mathbf{u}\cdot\nabla)\,\underline{\omega}}_{\text{advection}}=\underbrace{(\underline{\omega}\cdot\nabla)\,\mathbf{u}}_{\text{amplification}}+\underbrace{\nu\nabla^2\underline{\omega}}_{\text{diffusion}}.$$

Usually diffusion away from the boundary is balanced by advection towards the boundary, leading to vorticity confinement.

9. Kelvin-Helmholtz: velocity shear. Taylor-Rayleigh: density difference.

Set-up. $\mathbf{u}_{1,2}=(U_{1,2},0)+\nabla\varphi_{1,2}$.

Boundary conditions. $\partial_t\eta+(U+\partial_x\varphi)\,\partial_x\eta=\frac{\partial\varphi}{\partial y}$, $\rho\left(\partial_t\varphi+\frac{1}{2}u^2+\frac{p}{\rho}+gy\right)=f(t)$.

Linearised kinematic boundary condition. $\partial_t\eta+U_{1,2}\partial_x\eta=\frac{\partial\varphi_{1,2}}{\partial y}$ at $y=0$.

Linearised dynamic boundary condition. $\rho_1\left(\partial_t\varphi_1+U_1\partial_x\varphi_1+g\eta\right)=\rho_2\left(\partial_t\varphi_2+U_2\partial_x\varphi_2+g\eta\right)$ at $y=0$.

SP

1. Microcanonical ensemble. *NVE*-ensemble.

$dE = đQ - pdV = TdS - pdV$, $S(E) = k\log\Omega(E) \Rightarrow$

$$T = \left.\frac{\partial E}{\partial S}\right|_V,\ p = \left.\frac{\partial E}{\partial V}\right|_S = T\left.\frac{\partial S}{\partial V}\right|_E$$

$$C_V = \left.\frac{\partial E}{\partial T}\right|_V = T\left.\frac{\partial S}{\partial T}\right|_V,\ C_p = T\left.\frac{\partial S}{\partial T}\right|_p.$$

Canonical ensemble. *NVT*-ensemble.

$Z = \sum_n e^{-\beta E_n}$, $p(n) = \frac{e^{-\beta E_n}}{Z}$, $\partial_T = -k\beta^2\partial_\beta \Rightarrow$

$$\begin{aligned}
\langle E\rangle &= -\frac{\partial}{\partial\beta}\log Z\\
\Delta E^2 &= -\frac{\partial}{\partial\beta}\langle E\rangle = \frac{\partial^2}{\partial\beta^2}\log Z\\
S &= -k\sum_n p(n)\log p(n) = k\frac{\partial}{\partial T}(T\log Z)\\
C_V &= \frac{1}{kT^2}\frac{\partial^2}{\partial\beta^2}\log Z = \frac{1}{kT^2}\Delta E^2\\
F &:= E - TS = -kT\log Z.
\end{aligned}$$

Grand canonical ensemble. μVT-ensemble. μ is the energy cost to add a particle to a system while holding *S*, *V* constant.

$\mathcal{Z} = \sum_n e^{-\beta(E_n-\mu N_n)}$, $p(n) = \frac{e^{-\beta(E_n-\mu N_n)}}{\mathcal{Z}}$, $\partial_\beta = -kT^2\partial_T \Rightarrow$

$$\begin{aligned}
\langle E\rangle &= \mu\langle N\rangle - \frac{\partial}{\partial\beta}\log\mathcal{Z}\\
\langle N\rangle &= \frac{1}{\beta}\frac{\partial}{\partial\mu}\log\mathcal{Z}\\
\Delta N^2 &= \frac{1}{\beta}\frac{\partial}{\partial\beta}\langle N\rangle\\
S &= -k\sum_n p(n)\log p(n) = k\frac{\partial}{\partial T}(T\log\mathcal{Z})\\
\Phi &:= E - TS - \mu N = -p(T,\mu)V = -kT\log\mathcal{Z}.
\end{aligned}$$

In the thermodynamical limit $N\to\infty$, energy and particle number fluctuations are small and close to the average, so three ensembles coincide.

2. Boltzmann distribution. $p(n) = \frac{e^{-\frac{E_n}{kT}}}{\sum_m e^{-\frac{E_m}{kT}}}$. Maxwell distribution. $p(v) = \mathcal{N}v^2e^{-\frac{mv^2}{2kT}}$. Planck distribution. $p(\omega) = \mathcal{N}\frac{\omega^3}{e^{\frac{\hbar\omega}{kT}}-1}$. Bose-Einstein distribution. $\langle n_r\rangle = \frac{1}{e^{\beta(E_r-\mu)}-1}$. Fermi-Dirac distribution. $\langle n_r\rangle = \frac{1}{e^{\beta(E_r-\mu)}+1}$.

N.B. P: $Z_\omega = \sum_{n=0}^{\infty} e^{-\beta n\hbar\omega} = \frac{1}{1-e^{-\beta\hbar\omega}} \Rightarrow \ln Z = \int_0^\infty g(\omega)\ln Z_\omega d\omega$; B-E: $\mathcal{Z} = \prod_r \sum_{n_r} e^{-\beta n_r(E_r-\mu)}$ and F-D: $\mathcal{Z} = \prod_r \sum_{n=0,1} e^{-\beta n(E_r-\mu)}$.

3. $p = \frac{NkT}{V-bN} - a\frac{N^2}{V^2}$, *a* captures the effect of the attractive interaction at large distances. *b* is the effective reduction in the volume of the gas due to space occupied by the particles, arising from the repulsive potential.

4. High temperatures $\zeta \to 0$, low temperature $\zeta \to 1$. T_c is the temperature at which $\zeta = 1$.

5. $g_n(z) = \frac{1}{\Gamma(n)}\int_0^\infty dx \frac{x^{n-1}}{z^{-1}e^x-1} \equiv \sum_{m=1}^{\infty} \frac{z^m}{m^n}$. $g_n(1) = \zeta(n)$.

$$\int_0^\infty \frac{x^{n-1}}{z^{-1}e^x-1}dx = \frac{(\log z)^{n+1}}{n+1} + \frac{\pi^2}{6}n(\log z)^{n-1}.$$

6. Energy. $dE = TdS - pdV + \mu dN$ (first law).

Enthalpy. $H(S,p) = E + pV$.

Helmholtz free energy. $F(T,V) = E - TS$.

Grand canonical potential. $\Phi = E - TS - \mu N$.

Gibbs' free energy. $G(T,p,N) = E - TS + pV = \mu(T,p)N$.

7. An adiabatic process is one that occurs without transfer of heat or matter between a thermodynamic system and its surroundings. In an adiabatic process, energy is transferred only as work and $dS = 0$.

A quasi-static process is a thermodynamic process that happens slowly enough for the system to remain in internal equilibrium, i.e. infinitesimally close to eqlb.

A reversible process is one that is reversible without increasing entropy.

A Carnot cycle consists of: an isothermal expansion, an adiabatic expansion, an isothermal contraction and an adiabatic contraction.

8. The coexistence of liquid and gas in equilibrium requires the same pressure (mechanical), temperature (thermal) but further requires the same chemical potential (matter) $\mu_\text{liquid} = \mu_\text{gas} \Rightarrow G_\text{liquid} = G_\text{gas}$. Have $d\mu = \left.\frac{\partial\mu}{\partial p}\right|_T dp = \frac{1}{N}\left.\frac{\partial G}{\partial p}\right|_{N,T} dp = \frac{V(p,T)}{N}dp$ and integrate from p_liquid.

9. $m = \frac{1}{N}\sum_i \langle s_i\rangle = \frac{1}{N\beta}\frac{\partial \log Z}{\partial B}$ since $Z = \sum_{\{s_i\}} e^{-\beta E[s_i]}$, $E = -J\sum_{\langle ij\rangle} s_i s_j - B\sum_i s_i$.

Mean-field theory.

$$\sum_{\langle ij\rangle} s_i s_j = \sum_{\langle ij\rangle}\left[\underbrace{(s_i-m)(s_j-m)}_{\text{small when summed}} + m(s_i+s_j) - m^2\right] = qm\sum_i s_i - \frac{1}{2}Nqm^2$$

so $Z = e^{-\frac{1}{2}\beta JNqm^2}\left(\sum_{s_i=\pm1} e^{\beta B_\text{eff} s_i}\right)^N$ as if independent where $B_\text{eff} = B + Jqm$.

GR

1. $\Gamma^a_{\ bc} = \frac{1}{2}g^{ad}\left(g_{bd,c} + g_{cd,b} - g_{bc,d}\right)$.

$$\begin{aligned}
\nabla_a T_{bc} &= T_{bc,a} - \Gamma^d_{\ ab}T_{dc} - \Gamma^d_{\ ac}T_{bd} \\
\nabla_a T^b_{\ c} &= T^b_{\ c,a} + \Gamma^b_{\ ad}T^d_{\ c} - \Gamma^d_{\ ac}T^b_{\ d} \\
\nabla_a T^{bc} &= T^{bc}_{\ \ ,a} + \Gamma^b_{\ ad}T^{dc} + \Gamma^c_{\ ad}T^{bd}.
\end{aligned}$$

2. For a tensor, $\bar{T}^{\bar{a}_1\dots\bar{a}_n}_{\ \ \ \ \ \ \ \ \ \bar{b}_1\dots\bar{b}_m} = \frac{\partial \bar{x}^{\bar{a}_1}}{\partial x^{a_1}}\cdots\frac{\partial \bar{x}^{\bar{a}_n}}{\partial x^{a_n}}T^{a_1\dots a_n}_{\ \ \ \ \ \ \ \ \ b_1\dots b_m}\frac{\partial x^{b_1}}{\partial \bar{x}^{\bar{b}_1}}\cdots\frac{\partial x^{b_m}}{\partial \bar{x}^{\bar{b}_m}}$.

For the Levi-Civita connection, $\bar{\Gamma}^{\bar{a}}_{\ \bar{b}\bar{c}} = \frac{\partial x^b}{\partial \bar{x}^{\bar{b}}}\frac{\partial x^c}{\partial \bar{x}^{\bar{c}}}\Gamma^a_{\ bc}\frac{\partial \bar{x}^{\bar{a}}}{\partial x^a} + \frac{\partial^2 x^a}{\partial \bar{x}^{\bar{b}}\partial \bar{x}^{\bar{c}}}\frac{\partial \bar{x}^{\bar{a}}}{\partial x^a}$.

3. Recall $T^a = \frac{\partial x^a}{\partial \lambda}$, $V^a = \frac{\partial x^a}{\partial s}$.

1) $\nabla_V T^a = \nabla_T V^a = \frac{\partial^2 x^a}{\partial\lambda\partial s} + \Gamma^a_{\ bc}T^bV^c$;

2) $\nabla_{[V}\nabla_{T]}Q^a = \cancel{(\nabla_V T^b)(\nabla_b Q^a)} - \cancel{(\nabla_T V^b)(\nabla_b Q^a)} + T^bV^c\nabla_c\nabla_b Q^a - V^cT^b\nabla_b\nabla_c Q^a$;

3) $D_\tau^2 V^a = \nabla_T\nabla_T V^a = \nabla_T\nabla_V T^a = \cancelto{0}{\nabla_V\nabla_T T^a} - R^a_{\ bcd}T^bV^cT^d$ then relabel.

4. The Ricci identity. $2\nabla_{[a}\nabla_{b]}V^c = R^c_{\ dab}V^d$ and $2\nabla_{[a}\nabla_{b]}N_c = -R^d_{\ cab}N_d$ where Riemann tensor $R^c_{\ dab} = 2\Gamma^c_{\ d[b,a]} + \Gamma^c_{\ e[a}\Gamma^e_{\ b]d}$.

Symmetries. 1) $R_{(ab)cd} = R_{ab(cd)} = 0$; 2) $R_{abcd} = R_{cdab}$; 3) $R_{a[bcd]} = 0$ (Bianchi's first); 4) $R_{ab[cd;e]} = 0$ (Bianchi's [second]).

Proof:

1) By definition. By $0 = g_{ab;[cd]} = -R^e_{\ adc}g_{eb} - R^e_{\ bdc}g_{ea} = -2R_{(ab)cd}$.

2) Repeatedly use 1) and 3).

3) $0 = \phi_{;[ab]} \Rightarrow 0 = \phi_{;[abc]}$. But $\phi_{;a[bc]} = -\frac{1}{2}R^d_{\ acb}\phi_{,d}$ and ϕ is arbitrary.

4) In LIC, $\Gamma = 0$ so $R_{abcd;e} = R_{abcd,e}$ then use definition of R obtain $R_{abcd;e} = \Gamma^a_{\ bd,ce} - \Gamma^a_{\ bc,de}$.

EFE. $G_{ab} = \kappa T_{ab} - \Lambda g_{ab}$ where $\kappa = \frac{8\pi G}{c^4}$ and $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$ which is divergence free.

W

1. Acoustic velocity potential φ satisfies the wave equation with wave speed $c_0^2 = \left.\frac{dp}{d\rho}\right|_{\rho_0}$.

The linearised perturbation relations are

$$\begin{aligned}
\mathbf{u} &= \nabla\varphi \\
\tilde{p} &= \tilde{\rho}c_0^2 \\
\tilde{p} &= -\rho_0\frac{\partial\varphi}{\partial t} \\
\mathbf{I} &= \tilde{p}\mathbf{u}
\end{aligned}$$

where for a plane wave $\tilde{p} = \rho_0 c_0 \mathbf{u}\cdot\hat{\mathbf{k}}$.

The thermodynamic relations are

$$\begin{aligned} e &= \frac{1}{\gamma-1}\frac{p}{\rho} \\ c^2 &= \gamma\frac{p}{\rho} \\ \frac{p}{p_0} &= \left(\frac{\rho}{\rho_0}\right)^{\gamma}. \end{aligned}$$

The mechanical energy equations is

$$\frac{d}{dt}\left(\underbrace{\frac{1}{2}\rho_0\mathbf{u}^2}_{E_k} + \underbrace{\frac{1}{2}\frac{c_0^2}{\rho_0}\tilde{\rho}^2}_{E_p}\right) + \nabla.\mathbf{I} = 0.$$

2. If $\theta = \Theta e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$, $\phi = \Phi e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ then

$$\langle\theta\phi\rangle_t = \frac{1}{2}\mathrm{Re}\left(\Theta^*\Phi\right).$$

3. Gas. $\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+\rho\frac{\partial u}{\partial x} = 0$ (1), $\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{c^2}{\rho}\frac{\partial\rho}{\partial x} = 0$ (2). $\pm\frac{c}{\rho}(1)+(2) = 0$ using $Q = \int^{\rho}\frac{c(\rho')}{\rho'}d\rho'$. Shallow-water. $\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+h\frac{\partial u}{\partial x} = 0$ (3), $\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+g\frac{\partial h}{\partial x} = 0$ (4). $\pm\sqrt{\frac{g}{h}}(3)+(4) = 0$ using $Q = \int^{\rho}\frac{c(h')}{h'}dh'$, $c = \sqrt{gh}$.

4.

$$\begin{aligned} \rho_1 u_1 &= \rho_2 u_2 \\ p_1 + \rho_1 u_1^2 &= p_2 + \rho_2 u_2^2 \\ \frac{1}{2}u_1^2 + e_1 + \frac{p_1}{\rho_1} &= \frac{1}{2}u_2^2 + e_2 + \frac{p_2}{\rho_2} \end{aligned}$$

and with $u_{1,2} \mapsto u_{1,2} - V$.

$$\begin{aligned} h_1 u_1 &= h_2 u_2 \\ \frac{1}{2}gh_1^2 + h_1 u_1^2 &= \frac{1}{2}gh_2^2 + h_2 u_2^2. \end{aligned}$$

5. $\sigma_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij}$.

interface	boundary conditions
rigid	$\mathbf{u} = 0$
free	$\sigma\cdot\mathbf{n} = 0$
solid-fluid	$\mathbf{u}\cdot\mathbf{n}\rvert_-^+ = 0$, $\mathbf{n}\cdot\sigma\cdot\mathbf{n} = -p$, $\mathbf{n}\times\sigma\cdot\mathbf{n} = 0$
solid-solid	$\mathbf{u}\rvert_-^+ = 0$, $\sigma\cdot\mathbf{n}\rvert_-^+ = 0$

6. $c_s^2 = \frac{\mu}{\rho} < c_p^2 = \frac{\lambda+2\mu}{\rho}$. $\mathbf{u} = \nabla\phi + \nabla \times \underline{\psi}$ where $\nabla^2\phi - \frac{1}{c_p^2}\frac{\partial^2\phi}{\partial t^2} = 0$ and $\nabla^2\underline{\psi} - \frac{1}{c_s^2}\frac{\partial^2\underline{\psi}}{\partial t^2} = 0$.

1) P-wave: $\phi = Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$, $\mathbf{u} = i\mathbf{k}Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$;

2) S-wave: $\underline{\psi} = \mathbf{B}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$, SV $\mathbf{u} = i\left(\mathbf{k}\times\mathbf{B}\cdot\hat{\mathbf{z}}\right)\hat{\mathbf{z}}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ and SH $\mathbf{u} = i\left(\hat{\mathbf{z}}\cdot\mathbf{B}\right)\mathbf{k}\times\hat{\mathbf{z}}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$.

7. Equations.

$$\begin{aligned} \frac{D\rho}{Dt} &= 0 \\ \nabla\cdot\mathbf{u} &= 0 \\ \rho\frac{D\mathbf{u}}{Dt} &= -\nabla p - \rho g\hat{\mathbf{z}} \\ \frac{dp_0}{dz} &= -\rho g \end{aligned}$$

Perturbation. $\mathbf{u} = 0$, $\rho = \rho_0(z)$, $p = p_0 - \int \rho_0 g dz$.

Linearisation.

$$\begin{aligned} \frac{\partial\tilde{\rho}}{\partial t} + w\frac{d\rho_0}{dz} &= 0 \\ \nabla\cdot\mathbf{u} &= 0 \\ \rho\frac{\partial\mathbf{u}}{\partial t} &= -\nabla\tilde{p} - \tilde{\rho}g\hat{\mathbf{z}}. \end{aligned}$$

i.e. $(\rho_0 w_{zz} + \rho_0' w_z)_{tt} - \left(\partial_x^2 + \partial_y^2\right)\left(\rho_0' g - \rho_0\partial_t^2 w\right) = 0$

Slowly-varying assumption. ρ_0 and ρ_0' is effectively constant over a wavelength $\mathcal{O}(k^{-1})$.

Governing equation.

$$\left[\partial_t^2\nabla^2 + N^2\left(\partial_x^2 + \partial_y^2\right)\right]w = 0$$

where the Brunt-Väisälä frequency $N(z)^2 = -g\frac{\rho_0'(z)}{\rho_0(z)}$.

8. Define the local frequency and wavenumber $\omega = -\partial_t\theta$, $\mathbf{k} = \nabla\theta$, then given a local dispersion relation $\omega = \Omega(\mathbf{k};\, \mathbf{x},\, t)$, have

$$\frac{\partial\omega}{\partial x_i} = -\frac{\partial k_i}{\partial t},\quad \frac{\partial k_i}{\partial x_j} = \frac{\partial k_j}{\partial x_i}$$

so $\frac{\partial\omega}{\partial t} = \frac{\partial\Omega}{\partial t} - \left(\mathbf{c}_g\cdot\nabla\right)\omega$, $-\frac{\partial k_i}{\partial t} = \frac{\partial\omega}{\partial x_i} = \frac{\partial\Omega}{\partial x_i} + \mathbf{c}_g\cdot\frac{\partial\mathbf{k}}{\partial x_i} = \frac{\partial\Omega}{\partial x_i} + \mathbf{c}_g\cdot\frac{\partial k_i}{\partial\mathbf{x}}$, i.e.

$$\begin{aligned}\frac{d\mathbf{x}}{dt} &= \mathbf{c}_g \coloneqq \nabla_{\mathbf{k}}\Omega \\ \left.\frac{d}{dt}\right|_{\mathbf{c}_g} \omega &= \frac{\partial \Omega}{\partial t} \\ \left.\frac{d}{dt}\right|_{\mathbf{c}_g} \mathbf{k} &= -\nabla_{\mathbf{x}}\Omega \\ \left.\frac{d}{dt}\right|_{\mathbf{c}_g} &= \frac{\partial}{\partial t} + \mathbf{c}_g \cdot \nabla_{\mathbf{x}}.\end{aligned}$$

Fermat's principle is derived from the variational principle $\delta \int_{t_1}^{t_2} \Phi(\mathbf{x}, \dot{\mathbf{x}}, t; \mathbf{k}) dt = 0$ where $\Phi = \mathbf{k} \cdot \dot{\mathbf{x}} - \Omega(\mathbf{k}; \mathbf{x}, t)$. Snell's law follows when $\Omega = \Omega(k; z)$.

9. In the frame of the source moving at $\mathbf{U}$ relative to the fluid, $\mathbf{X} = \mathbf{x} - \mathbf{U}t \Rightarrow \partial_t|_{\mathbf{x}} \mapsto \partial_t|_{\mathbf{X}} - \mathbf{U} \cdot \partial_{\mathbf{X}}$ so $-i\omega_r = -i\omega_s - \mathbf{U} \cdot i\mathbf{k}$, i.e. $\omega_s = \omega_r - \mathbf{U} \cdot \mathbf{k}$.